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多種 Lotka-Volterra 非自励競争モデルの解の漸近的性質

Asymptotic property of solutions of nonautonomous Lotka-Volterra model for N-competing species

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1 Introduction and Statements of the main results

In this paper we consider the system of differential equations

$$u_i' = u_i \left[a_i(t) - \sum_{j=1}^{N} b_{ij}(t) f_{ij}(u_i,\ u_j) \right], \quad i = 1, \dots, N,\ N \geq 2, \tag{GLV}$$

where the functions $a_i(t)$, $1 \leq i \leq N$, and $b_{ij}(t)$, $1 \leq i,j \leq N$, are assumed to be continuous and nonnegative on $\mathbb{R}$. Furthermore, let the functions $f_{ij}(x,\ y)$, $1 \leq i,\ j \leq N$, be continuously differentiable on $\mathbb{R}_+^2 = (0,\ \infty)^2$, and we impose the following conditions on f_{ij}'s:

$$\begin{cases} f_{ii}(x,\ y),\ 1 \leq i \leq N,\ \textit{is continuously differentiable on}\ [0,\ \infty) \times [0,\ \infty); \\ f_{ij}(x,\ y) > 0, \quad (x,\ y) \in \mathbb{R}_+^2,\ 1 \leq i,\ j \leq N; \\ (D_1 f_{ii} + D_2 f_{ii})(x,\ x) > 0, \quad x \in \mathbb{R}_+,\ 1 \leq i \leq N; \\ D_1 f_{ij}(x,\ y) \geq 0, \quad (x,\ y) \in \mathbb{R}_+^2,\ 1 \leq i,\ j \leq N; \\ D_2 f_{ij}(x,\ y) \geq 0, \quad (x,\ y) \in \mathbb{R}_+^2,\ 1 \leq i,\ j \leq N; \\ f_{ii}(0,\ 0) = 0, \quad 1 \leq i \leq N; \\ \lim\limits_{x \to \infty} f_{ii}(x,\ x) = \infty, \quad 1 \leq i \leq N, \end{cases} \tag{1.1}$$

where D_i, $i = 1,\ 2$, denotes the differentiation with respect to the i-th variable.

System (GLV) is a generalization of the following nonautonomous N-dimensional Lotka-Volterra competition system which S. Ahmad and A. C. Lazer [2] considered:

$$u_i' = u_i \left[a_i(t) - \sum_{j=1}^{N} b_{ij}(t) u_j \right], \quad i = 1, \dots, N,\ N \geq 2. \tag{LV}$$

An prototype of system (LV), as well as (GLV), is the classical Lotka-Volterra competition model for two species:

$$\begin{cases} u_1' = u_1(a_1 - b_{11}u_1 - b_{12}u_2), \\ u_2' = u_2(a_2 - b_{21}u_1 - b_{22}u_2), \end{cases} \tag{1.2}$$

where a_i, $i = 1,\ 2$, and b_{ij}, $i,\ j = 1,\ 2$, are positive constants. When the growth rates a_i, $i = 1,\ 2$, and the interaction coefficients b_{ij}, $i, j = 1,\ 2$, satisfy

$$a_1 - b_{12} \left(\frac{a_2}{b_{22}} \right) > 0, \quad a_2 - b_{21} \left(\frac{a_1}{b_{11}} \right) > 0, \tag{1.3}$$

there exists a unique equilibrium point $(u_1^*, u_2^*) \in \mathbb{R}_+^2$. It is known that, if (1.3) hold, then any solution $(u_1(t), u_2(t))$ of system (1.2) with $(u_1(t_0), u_2(t_0)) \in \mathbb{R}_+^2$ satisfies

$$u_1(t) \to u_1^* \quad \text{and} \quad u_2(t) \to u_2^* \quad \text{as } t \to \infty.$$

In [2]–[4] it is shown that analogous results still hold for the nonautonomous equation (LV), as seen below. In this paper we intend to generalize such results further.

We introduce notation. Put $c_M := \sup_{t\in\mathbb{R}} c(t)$ for bounded functions $c(t)$ on $\mathbb{R}$. For $i = 1, \ldots, N$, we put

$$\tilde{f}_{ii}(x) = f_{ii}(x, x), \quad x \in \mathbb{R}_+.$$

By assumption (1.1) $\tilde{f}_{ii}$, $i = 1, \ldots, N$, have the inverse function $\tilde{f}_{ii}^{-1} : \mathbb{R}_+ \to \mathbb{R}_+$. The assumptions employed in the paper will be sellected from the following list:

(A1) $b_{ii}(t) > 0, \quad t \in \mathbb{R},\ 1 \le i \le N;$

(A2) $\displaystyle\int_0^\infty b_{ii}(s)ds = \infty, \quad 1 \le i \le N;$

(A3) $\displaystyle\left(\frac{a_i}{b_{ii}}\right)_M < \infty, \quad 1 \le i \le N;$

(A4) $\displaystyle\inf_{t\in\mathbb{R}} \frac{a_i(t) - \sum_{j\neq i} b_{ij}(t)(a_j/b_{jj})_M}{b_{ii}(t)} > 0, \quad 1 \le i \le N;$

(A5) $\displaystyle\inf_{t\in\mathbb{R}} \frac{a_i(t) - \sum_{j\neq i} b_{ij}(t) f_{ij}(\tilde{f}_{ii}^{-1}((a_i/b_{ii})_M),\ \tilde{f}_{jj}^{-1}((a_j/b_{jj})_M))}{b_{ii}(t)} > 0, \ 1 \le i \le N;$

(A6) $f_{ij}(x, y) \le \tilde{f}_{jj}(y), \quad (x, y) \in \mathbb{R}_+^2,\ 1 \le i, j \le N;$

(A7) *for any $s > 1$ sufficientry close to 1;*
$f_{ij}(\tilde{f}_{ii}^{-1}(sx), \tilde{f}_{jj}^{-1}(sy)) \le s f_{ij}(\tilde{f}_{ii}^{-1}(x), \tilde{f}_{jj}^{-1}(y)),\ (x, y) \in \mathbb{R}_+^2,\ 1 \le i, j \le N.$

REMARK 1.1. As in the case of (LV) and (1.2), if $f_{ij}(x, y)$, $1 \le i, j \le N$, are independent of x, (A6) is satisfied. For (LV) we can take $f_{ij}(x, y) = y$, $1 \le i, j \le N$, which satisfy (A6) and (A7).

REMARK 1.2. Let

$$f_{ij}(x, y) = \begin{cases} \dfrac{x^{\alpha_{ij}}}{1 + x^{\alpha_{ij}}} y^{\beta_{ij}}, & i \neq j, \\ x^{\alpha_{ij}} y^{\beta_{ij}}, & i = j, \end{cases}$$

where $\alpha_{ij}, \beta_{ij} \in \mathbb{R}_+$. If for $i \neq j$, $\beta_{ij} = \alpha_{jj} + \beta_{jj}$, then the functions f_{ij}, $1 \le i, j \le N$, satisfy (A6).

REMARK 1.3. Let

$$f_{ij}(x, y) = x^{\alpha_{ij}} y^{\beta_{ij}}, \quad (x, y) \in \mathbb{R}^2,\ 1 \le i, j \le N,$$

where $\alpha_{ij}, \beta_{ij} \in \mathbb{R}_+$. If $\alpha_{ij} + \beta_{ij} \le \min\{\alpha_{ii} + \beta_{ii},\ \alpha_{jj} + \beta_{jj}\}$, then the functions f_{ij}, $1 \le i, j \le N$, satisfy (A7).

S. Ahmad and A. C. Lazer [2] supposed that the functions $a_i(t)$, $1 \le i \le N$ and $b_{ij}(t)$, $1 \le i, j \le N$, satisfy conditions (A1)–(A3) and (A4). Under these conditions they have shown the following [2]:

(I) *If $u = (u_i, \ldots, u_N)$ is a solution of* (LV) *with $u_i(t_0) > 0$, $1 \le i \le N$, $t_0 \in \mathbb{R}$, then*

$$0 < \inf_{t \ge t_0} u_i(t) \le \sup_{t \ge t_0} u_i(t) < \infty, \quad \textit{for } 1 \le i \le N.$$

(II) *If A is a compact subset of $\mathbb{R}_+^N$, then the Lebesgue measure of the set $\{u(t) \mid u$ is a solution of (LV) satisfying $u(t_0) \in A\}$ tends to 0 as $t \to \infty$.*

Our main aim is to show that (I) and (II) are still valid for (GLV). To state the results we introduce the symbol: For compact subset A of $\mathbb{R}_+^N$ and $t_0 \in \mathbb{R}$ we set

$$u(t,\ t_0,\ A) = \{u(t) \mid u \text{ is a solution of (GLV) satisfying } u(t_0) \in A\}.$$

By $\mu(\cdot)$ we denote the Lebesgue measure of measurable sets in $\mathbb{R}_+^N$. We can show the following:

THEOREM 1.4. *Let conditions* (A1)–(A3), (A4), *and* (A6) *hold. Let A be a compact subset of $\mathbb{R}_+^N$ and let $t_0 \in \mathbb{R}$. Then,*

$$\mu(u(t,\ t_0,\ A)) \to 0 \quad \text{as } t \to \infty.$$

THEOREM 1.5. *Let conditions* (A1)–(A3), (A5), and (A7) *hold. Let A be a compact subset of $\mathbb{R}_+^N$ and let $t_0 \in \mathbb{R}$. Then,*

$$\mu(u(t,\ t_0,\ A)) \to 0 \quad \text{as } t \to \infty.$$

We give examples of systems (GLV) for which above conditions hold.

EXAMPLE 1.6. We consider system (GLV) for two species

$$u_1' = u_1\left[(\cos t + 7) - (\sin t + 7)\cdot u_1^2 - (\sin t + 1)\cdot\left(\frac{u_1^3}{1+u_1^3}\cdot u_2^2\right)\right],$$

$$u_2' = u_2\left[(\cos t + 9) - (\sin t + 2)\cdot\left(\frac{u_2^4}{1+u_2^4}\cdot u_1^3\right) - (\sin t + 9)\cdot u_2^3\right].$$

Obviously (A6) holds. We have

$$a_1(t) - b_{12}(t)\left(\frac{a_2}{b_{22}}\right)_M > \cos t + 7 - (\sin t + 1)\cdot\frac{10}{8} > 2,$$

$$a_2(t) - b_{21}(t)\left(\frac{a_1}{b_{11}}\right)_M > \cos t + 9 - (\sin t + 2)\cdot\frac{8}{6} > 2.$$

So conditions (A1)–(A3) and (A4) hold. Of course condition (1.1) hold.

EXAMPLE 1.7. We consider system (GLV) for two-species

$$u_1' = u_1[(\cos t + 7) - (\sin t + 7)\cdot u_1^4 - (\sin t + 1)\cdot u_1 u_2^2],$$

$$u_2' = u_2\left[(\cos t + 9) - (\sin t + 2)\cdot u_2^2 u_1^2 - (\sin t + 9)\cdot u_2^6\right].$$

Obviously (A7) holds. We have

$$\begin{aligned} a_1(t) - b_{12}(t) f_{12}&\left(\tilde{f}_{11}^{-1}\left(\left(\frac{a_1}{b_{11}}\right)_M\right),\ \tilde{f}_{22}^{-1}\left(\left(\frac{a_2}{b_{22}}\right)_M\right)\right) \\ &> \cos t + 7 - (\sin t + 1)\cdot\left(\frac{8}{6}\right)^{1/4}\cdot\left(\frac{10}{8}\right)^{2/6} > 2, \\ a_2(t) - b_{21}(t) f_{21}&\left(\tilde{f}_{22}^{-1}\left(\left(\frac{a_2}{b_{22}}\right)_M\right),\ \tilde{f}_{11}^{-1}\left(\left(\frac{a_1}{b_{11}}\right)_M\right)\right) \\ &> \cos t + 9 - (\sin t + 2)\cdot\left(\frac{10}{8}\right)^{2/6}\cdot\left(\frac{4}{3}\right)^{2/4} > 2. \end{aligned}$$

So conditions (A1)–(A3), (A5) hold. Of course condition (1.1) hold.

2 The sketch of the proof of the main results

In this section we give the sketch of the proof of the main results. As a first step, we note that every solutions u of (GLV) with $u(t_0) \in \mathbb{R}_+^N$ remains here as long as it exists. To see this we rewrite system (GLV) in the form

$$u_i'(t) = p_i(t)u_i(t), \quad i = 1, 2, \dots, N,$$

where the functions $p_i(t)$, $1 \le i \le N$, are given by

$$p_i(t) = a_i(t) - \sum_{j=1}^{N} b_{ij}(t) f_{ij}(u_i(t), u_j(t)).$$

Since p_i, $1 \le i \le N$, is continuous on the domain of u, for t in the domain of u we obtain

$$u_i(t) = u_i(t_0) \exp \int_{t_0}^{t} p_i(s) ds > 0.$$

Hence $u(t) \in \mathbb{R}_+^N$. Next we rewrite system (GLV) in the form

$$u' = g(u, t),$$

where $u(t) = (u_1(t), \dots, u_N(t)) \in \mathbb{R}^N$, and $g(u, t) = (g_1(u, t), \dots, g_N(u, t))$ is given by

$$g_i(x, t) = x_i \left[a_i(t) - \sum_{j=1}^{N} b_{ij}(t) f_{ij}(x_i, x_j) \right], \quad 1 \le i \le N,$$

for $x = (x_1, \dots, x_N) \in \mathbb{R}^N$. Since the functions a_i, $1 \le i \le N$, and b_{ij}, $1 \le i, j \le N$, are continuous on $\mathbb{R}$ and the functions f_{ij}, $1 \le i, j \le N$, are continuousuly differentiable on $\mathbb{R}_+^2$, for every $\xi = (\xi_i) \in \mathbb{R}_+^N$ and $\tau \in \mathbb{R}$, there exists a unique solution $u(t)$ of (GLV) with $u(\tau) = \xi$. We denote it by $u(t, \tau, \xi) = (u_i(t, \tau, \xi))$. Recall that we have introduced the notation:

$$u(t, t_0, A) = \{u(t, t_0, \xi) \mid \xi \in A\}$$

for $A \subset \mathbb{R}_+^N$. Furthermore, since the functions $g_i(x, t)$, $1 \le i \le N$, are continuously differentiable with respect to the components of $x \in \mathbb{R}^N$, $u(t, \tau, \xi)$ are continuously differentiable with respect to the components of $\xi \in \mathbb{R}^N$. Therefore we can introduce the following notations. We denote by $D_\xi(u(t, \tau, \xi))$ the $N \times N$ matrix with (i, j)th entry equal to $\partial u_i(t, \tau, \xi)/\partial \xi_j$:

$$D_\xi u(t, \tau, \xi) = \left[\frac{\partial u_i(t, \tau, \xi)}{\partial \xi_j} \right],$$

where $\xi \in \mathbb{R}_+^N$. Similarly we define $N \times N$ matrix $D_x g(x, t)$ by

$$D_x g(x, t) = \left[\frac{\partial g_i(x, t)}{\partial x_j} \right],$$

where $x \in \mathbb{R}_+^N$.

Now for $t \ge t_0$ and $\xi_0 \in \mathbb{R}_+^N$, we set $u_0(t) = u(t, t_0, \xi_0)$. Then it is well known [6] that

$$X'(t) = A(t)X(t), \quad X(t_0) = I,$$

where

$$X(t) = D_\xi u(t, t_0, \xi_0), \quad A(t) = D_x g(u_0(t), t),$$

and I is the $N \times N$ identity matrix. Furthermore we know that

$$\det X(t) = \exp \int_{t_0}^{t} \mathrm{tr} A(s) ds.$$

Therefore, we have

$$\det D_\xi u(t,\ t_0,\ \xi_0) = \exp \int_{t_0}^{t} \sum_{i=1}^{N} \frac{\partial g_i}{\partial x_i}(u_0(s),\ s)ds.$$

Hence it follows from the change of variables formula that

$$\begin{aligned}\mu(u(t,\ t_0,\ A)) &= \int_{u(t,\ t_0,\ A)} dx = \int_A \det D_\xi u(t,\ t_0,\ \xi_0) d\xi_0 = \int_A \exp \int_{t_0}^{t} \sum_{i=1}^{N} \frac{\partial g_i}{\partial x_i}(u_0(s),\ s) ds d\xi_0 \\ &\le \int_A \exp \left[\sum_{i=1}^{N} \log \frac{u_i(t)}{u_i(t_0)} - \int_{t_0}^{t} \sum_{i=1}^{N} b_{ii}(s) u_i(s) \tilde{f}'_{ii}(u_i(s)) ds \right] d\xi_0.\end{aligned}$$

Therefore, by (A2) and (1.1), in order to prove Theorems 1.4 and 1.5, it is sufficient to prove the following claim:

Claim (*see Taniguchi* [1, *Lemmas* 3.1 *and* 4.1]). *If either conditions* (A1)–(A3), (A4), *and* (A6) *or conditions* (A1)–(A3), (A5) *and* (A7) *hold, there exists some numbers* M_A, $\delta_A > 0$ *and* $t_A \ge t_0$ *such that for* $t \ge t_A$, $i = 1, \ldots, N$, *and* $\xi_0 \in A$,

$$\delta_A \le u_i(t,\ t_0,\ \xi_0) \le M_A. \tag{2.1}$$

In fact, by (1.1), (2.1), we have

$$\int_{t_0}^{t} \sum_{i=1}^{N} \frac{\partial g_i}{\partial x_i}(u_0(s),\ s) ds \le \sum_{i=1}^{N} \log \frac{M_A}{\delta_A} - \delta_A \delta'_A \int_{t_0}^{t} \sum_{i=1}^{N} b_{ii}(s) ds = N \log \frac{M_A}{\delta_A} - \delta_A \delta'_A \int_{t_0}^{t} \sum_{i=1}^{N} b_{ii}(s) ds,$$

where $\delta'_A := \min\{\tilde{f}_{ii}(\delta_A) \mid 1 \le i \le N\}$. Therefore, by (A2), we have

$$\int_{t_0}^{t} \sum_{i=1}^{N} \frac{\partial g_i}{\partial x_i}(u_0(s),\ s) ds \to -\infty \quad \text{as } t \to \infty$$

uniformly with respect to $\xi_0 \in A$; that is

$$\mu(u(t,\ t_0,\ A)) \to 0 \quad \text{as } t \to \infty.$$

This completes the proof.

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